1. Let (x_n) be a sequence of real numbers.

Define
$$S_n := \frac{x_1 + \cdots + x_n}{n} \quad \forall\, n \in \mathbb{N}.$$

(a) If $\lim(x_n) = 0 \in \mathbb{R}$, show that $\lim(S_n) = 0$.

(b) Is the converse of (a) true?

Proof of (a):

Let $$S_n = \frac{x_1 + \cdots + x_m}{n} + \frac{x_{m+1} + \cdots + x_n}{n}$$

Since (x_n) is convergent, hence it is bounded, i.e.

$|x_n| \le M \ \forall\, n \in \mathbb{N}$ for some $M > 0$.

Let $\varepsilon > 0$ be given.

Since $\lim(x_n) = 0$, $|x_n| < \frac{\varepsilon}{2} \ \forall\, n \geq m$.

By Archimedean Property, choose $N \in \mathbb{N}$ s.t.

$$N > \max\left\{ \frac{mM}{\varepsilon/2},\ m \right\}.$$

Then, for $n \geq N$, we have

$$|S_n| \leq \frac{|x_1| + \cdots + |x_m|}{n} + \frac{|x_{m+1}| + \cdots + |x_n|}{n}$$

$$\leq \frac{mM}{n} + \frac{(n-m)\frac{\varepsilon}{2}}{n}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

Hence, $\lim(S_n) = 0$.

(b) No. Counter-example:

$x_n := (-1)^n$ is divergent, while

$$S_n = \begin{cases} -\frac{1}{n}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$ converges to 0.

Recap:

<u>Monotone Convergence Theorem</u>:

A monotone sequence of $\mathbb{R}$ is convergent if and only if it is bounded.

(a) If (x_n) is increasing & bounded, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}.$$

(b) If (x_n) is decreasing & bounded, then

$$\lim(x_n) = \inf\{x_n : n \in \mathbb{N}\}.$$

Remark:

- Equivalent to Existence of Least upper bound Property. Hence could be viewed as a characterization of completeness of $\mathbb{R}$.
- Also equivalent to Bolzano-Weierstrass Theorem.

2. Let (x_n) be the sequence defined by

$$x_1 = 10, \quad x_{n+1} = \frac{2x_n}{x_n^2+1} \quad \forall n \geq 1.$$

Show that (x_n) is convergent & find its limit.

Proof:

(Boundedness): Note that, for $n \geq 1$,

$$1 - x_{n+1} = 1 - \frac{2x_n}{x_n^2+1} = \frac{1-2x_n+x_n^2}{x_n^2+1} = \frac{(x_n-1)^2}{x_n^2+1} \geq 0$$

By Mathematical induction, we have $x_n \geq 0 \ \forall n \in \mathbb{N}$.

Hence, $0 \leq x_n \leq 1 \ \forall n \geq 2$.

(Monotonicity): Check: $x_2 < \frac{20}{101} < \frac{2 \times \left(\frac{20}{101}\right)}{\left(\frac{20}{101}\right)^2+1} = x_3$.

Assume $x_k \leq x_{k+1}$ for some k.

Note that $x_{k+2} - x_{k+1} = \frac{2x_{k+1}}{x_{k+1}^2+1} - \frac{2x_k}{x_k^2+1}$

$$= \frac{2x_{k+1}x_k^2 + 2x_{k+1} - 2x_k x_{k+1}^2 - 2x_k}{(x_k^2+1)(x_2^2+1)}$$

$$= \frac{2(x_{k+1}-x_k)(1-x_k x_{k+1})}{(x_k^2+1)(x_2^2+1)} \geq 0.$$

Hence $x_{k+2} \geq x_{k+1}$.

By Mathematical induction, we have

$$x_{n+1} \geq x_n \quad \forall n \geq 2$$

Hence $(x_n)_{n\geq 2}$ is increasing & bounded, by MCT,

(x_n) is convergent. Let $\lim(x_n) = L$.

Hence

$$L = \frac{2L}{L^2+1}$$

$$\Rightarrow L(L-1)(L+1) = 0.$$

$$\Rightarrow L = 0 \text{ or } -1 \text{ or } 1.$$

(rejected, since $x_2 > 0$)

Therefore, $\lim(x_n) = 1$.

Exercise:

Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) s.t. $\lim\left(\frac{1}{x_{n_k}}\right) = 0$.

(Hint: $\forall k \in \mathbb{N}$, choose n_k s.t. $x_{n_k} > k$).